ON CONGRUENCES WITH PRODUCTS OF VARIABLES FROM SHORT INTERVALS AND APPLICATIONS

JEAN BOURGAIN, MOUBARIZ Z. GARAEV, SERGEI V. KONYAGIN, AND IGOR E. SHPARLINSKI

Abstract. We obtain upper bounds on the number of solutions to congruences of the type

$$(x_1+s)\dots(x_\nu+s)\equiv(y_1+s)\dots(y_\nu+s)\not\equiv 0 \pmod p$$

modulo a prime p with variables from some short intervals. We give some applications of our results and in particular improve several recent estimates of J. Cilleruelo and M. Z. Garaev on exponential congruences and on cardinalities of products of short intervals, some double character sum estimates of J. Friedlander and H. Iwaniec and some results of M.-C. Chang and A. A. Karatsuba on character sums twisted with the divisor function.

1. Introduction

For a prime p, let $\mathbb{F}_p$ be the field of residues modulo p. Also, denote $\mathbb{F}_p^* = \mathbb{F}_p\backslash\{0\}$. For integers h and $\nu \ge 1$ and elements $s \in \mathbb{F}_p$ and $\lambda \in \mathbb{F}_p^*$, we denote by $J_\nu(p,h,s;\lambda)$ the number of solutions of the congruence

$$\begin{aligned}(x_1+s)\dots(x_\nu+s)\equiv&\lambda \pmod p,\\ 1\le x_1,\dots,&x_\nu\le h.\end{aligned}\tag{1}$$

For large values of h, one can use bounds of Kloosterman sums (for $\nu = 2,3$) and multiplicative character sums (for $\nu \ge 4$) to obtain various asymptotic formulas for $J_\nu(p,h,s;\lambda)$, see [14, 15, 21, 24, 25]. However, this approach does not give any nontrivial estimates for small values of h, and thus Chan and Shparlinski [5], for $\nu = 2$, have employed methods of additive combinatorics, namely some results of Bourgain [3], in order to obtain a nontrivial upper bound on $J_\nu(p,h,s;\lambda)$ for any h.

Cilleruelo and Garaev [9] have substantially improved the bounds of [5], obtained several results for $\nu = 3$ and also suggested several conjectures.

Recently, motivated by some applications to certain algorithmic problems, new results on $J_\nu(p,h,s;\lambda)$ have been given by Bourgain, Garaev,

Konyagin and Shparlinski [4]. In particular, it is shown in [4] that for

$$h < p^{1/(\nu^2-1)},$$

we have the bound

$$J_\nu(p,h,s;\lambda) < \exp\left(c(\nu)\frac{\log h}{\log\log h}\right), \tag{2}$$

uniformly over $s \in \mathbb{F}_p$ and $\lambda \in \mathbb{F}_p^*$, where $c(\nu)$ depends only on $\nu = 2,3,\ldots$. In particular, for $\nu = 4$ the bound (2) answers the open question from [9, Section 6].

Here we use and develop further some ideas of [4] and study a symmetric version of the congruence (1). More precisely, for a prime p, integers h and $\nu \ge 1$ and an element $s \in \mathbb{F}_p$, we study the number of solutions $K_\nu(p,h,s)$ of the congruence

$$\begin{aligned}(x_1+s)\ldots(x_\nu+s) \equiv &(y_1+s)\ldots(y_\nu+s) \not\equiv 0 \pmod p,\\ 1 \le x_1,\ldots,&x_\nu,y_1,\ldots,y_\nu \le h.\end{aligned} \tag{3}$$

We note that for $\nu = 2$ this question, and its generalizations to residue rings and arbitrary finite fields, has been considered in a number of works [1, 10, 12, 20]. So, although our argument works for $\nu = 2$ as well, here we concentrate on the case $\nu \ge 3$.

We believe that our results are of independent interest and then may also be used to improve some previous results. For example, Corollary 20 extends the range of h under which a similar result is obtained in [4].

Furthermore, it is easy to see that bounds on $K_\nu(p,h,s)$ can be reformulated as statements about moments of character sums over the intervals $[s,s+h]$, for example, see Lemma 4 below. As such, they also complement various other results of the type which can be found in the literature, see [1, 10, 11, 12] and references therein. Using the ideas behind our estimates of $K_\nu(p,h,s)$ we estimate the number of solutions of several other congruences of similar form which in turn leads to improvements of the bounds

- of Cilleruelo and Garaev [9, Corollary 3] on the number of solutions to exponential congruences in small intervals;
- of Friedlander and Iwaniec [13] on double character sums over subsets of intervals;
- of Chang [7] and Karatsuba [17, 18] on the character sums with the divisor function.

2. Resultant Bound

For positive integers m, n with $m, n \geq 2$ and $\sigma \in \mathbb{R}$, we define the $(m+n-2) \times (n-1)$ circulant matrix $A(m, n, \sigma)$ as follows:

$$\begin{pmatrix} \sigma & \sigma+1 & \dots & \sigma+m-1 & 0 & 0 & \dots & 0 \\ 0 & \sigma & \dots & \sigma+m-2 & \sigma+m-1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \sigma & \sigma+1 & \dots & \dots & \sigma+m-1 \end{pmatrix}.$$

We mark all elements located in the intersection of i-th row and j-th column if $i \leq j \leq i+m-1$. Note that all unmarked elements are zeros and, conversely, for $\sigma > 0$ all zeros are unmarked.

Lemma 1. *Let $m, n \geq 2$ be integers and $\sigma, \vartheta \in \mathbb{R}$. If in the $(m+n-2) \times (m+n-2)$ matrix*

$$X(m,n) = \begin{pmatrix} A(m,n,\sigma) \\ A(n,m,\vartheta) \end{pmatrix}$$

we select $m+n-2$ marked elements such that each row and each column contains exactly one selected element then the sum of the selected elements is always equal to

$$\Sigma(m,n,\sigma,\vartheta) = (m-1+\sigma)(n-1+\vartheta) - \sigma\vartheta.$$

Proof. Let

$$X(m,n) = (x_{i,j})_{1 \leq i,j \leq m+n-2},$$

where i indicates the row. Since the sum of the diagonal elements of $X(m,n)$ is equal to $(m-1+\sigma)(n-1+\vartheta) - \sigma\vartheta$, it suffices to prove that the sum of the selected elements does not depend on the choice of selection. To see this, we transform the matrix $X(m,n)$ into a matrix

$$Y(m,n) = (y_{i,j})_{1 \leq i,j \leq m+n-2}$$

as follows

- If $x_{i,j}$ is unmarked, then we put $y_{i,j} = 0$
- If $x_{i,j}$ is marked, then we put

$$y_{i,j} = \begin{cases} x_{i,j} + 2i - \sigma, & \text{for } 1 \leq i \leq n-1, \\ x_{i,j} + 2i - n + 1 - \vartheta, & \text{for } n \leq i \leq m+n-2. \end{cases}$$

Since the selected elements occur in each row exactly once, from this transformation of $X(m,n)$ into $Y(m,n)$ the sum of the elements at the marked positions changes only by

$$\sigma_1 = \sum_{i=1}^{n-1} (2i - \sigma) + \sum_{i=n}^{m+n-2} (2i - n + 1 - \vartheta)$$

and in particular does not depend on the choice of the selection. Therefore, it suffices to show that the sum of corresponding selected elements of $Y(m,n)$ does not depend on the choice of selection. But this follows from the observation that when x_{ij} is marked, we have that

$$y_{i,j} = i + j.$$

Hence, the sum of the corresponding selected elements of $Y(m,n)$ is equal to

$$\sigma_2 = 2(1 + \ldots + (m+n-2)) = (m+n-1)(m+n-2)$$

and does not depend on the choice of selection. Since $\sigma_2 - \sigma_1 = \sigma$, the result now follows. $\square$

We need the following simple statement.

Lemma 2. *Let $M \ge m \ge 2$, $N \ge n \ge 2$ be integers, $\sigma + M - m \ge 0$, $\vartheta + N - n \ge 0$. Assume also that one of the following conditions hold:*

(i) $\sigma \ge 0$;
(ii) $\vartheta \ge 0$;
(iii) $\sigma + \vartheta \ge -1$.

Then $\Sigma(M,N,\sigma,\vartheta) \ge \Sigma(m,n,\sigma+M-m,\vartheta+N-n)$.

Proof. Clearly,

$$\begin{aligned} &\Sigma(M,N,\sigma,\vartheta) - \Sigma(m,n,\sigma+M-m,\vartheta+N-n) \\ &= (\sigma+M-m)(\vartheta+N-n) - \sigma\vartheta \ge 0. \end{aligned}$$

Since either of the conditions (i)–(iii) implies

$$(\sigma+M-m)(\vartheta+N-n) \ge \sigma\vartheta,$$

the result follows. $\square$

Corollary 3. *Let $H \ge 1$, $\sigma, \vartheta \in \mathbb{R}$, and let $M, N \ge 2$ be fixed integers. Assume that either of the conditions (i)–(iii) of Lemma 2 is satisfied. Let $P_1(Z)$ and $P_2(Z)$ be non-constant polynomials,*

$$P_1(Z) = \sum_{i=0}^{M-1} a_i Z^{M-1-i} \qquad \text{and} \qquad P_2(Z) = \sum_{i=0}^{N-1} b_i Z^{N-1-i}$$

such that

$$\begin{aligned} |a_i| &< H^{i+\sigma}, \quad i = 0, \ldots, M-1, \\ |b_i| &< H^{i+\vartheta}, \quad i = 0, \ldots, N-1. \end{aligned}$$

Then

$$\mathrm{Res}(P_1, P_2) \ll H^{\Sigma(M,N,\sigma,\vartheta)},$$

where the implicit constant in $\ll$ depends only on M and N.

Proof. Let $m-1=\deg P_1$ and $n-1=\deg P_2$. We have $2\le m\le M$, $2\le n\le N$. The inequalities $|a_{M-m}|\ge 1$ and $|b_{N-n}|\ge 1$ imply $\sigma+M-m\ge 0$ and $\vartheta+N-n\ge 0$, respectively. We recall that

$$\mathrm{Res}(P_1,P_2)=\det\begin{pmatrix} A\\ B\end{pmatrix},$$

where

$$A=\begin{pmatrix} a_{M-m} & \dots & a_{M-2} & a_{M-1} & 0 & 0 & \dots & 0\\ 0 & a_{M-m} & \dots & a_{M-2} & a_{M-1} & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & a_{M-m} & \dots & \dots & a_{M-2} & a_{M-1}\end{pmatrix}$$

and

$$B=\begin{pmatrix} b_{N-n} & \dots & b_{N-2} & b_{N-1} & 0 & 0 & \dots & 0\\ 0 & b_{N-n} & \dots & b_{N-2} & b_{N-1} & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & b_{N-n} & \dots & \dots & b_{N-2} & b_{N-1}\end{pmatrix}$$

are $(m+n-2)\times(n-1)$ and $(m+n-2)\times(m-1)$ matrices, respectively. The result now follows from the representation of the determinant by sums of products of its elements and Lemmas 1 and 2. $\square$

3. More General Congruences

To estimate $K_\nu(p,h,s)$ we sometimes have to study a more general congruence. For a prime p, integers h and $\nu\ge 1$ and a vector $\mathbf{s}=(s_1,\dots,s_\nu)\in\mathbb{F}_p$ we denote by $K_\nu(p,h,\mathbf{s})$ the number of solutions of the congruence

$$\begin{aligned}(x_1+s_1)\dots(x_\nu+s_\nu)\equiv&(y_1+s_1)\dots(y_\nu+s_\nu)\not\equiv 0 \pmod p,\\ 1\le x_1,\dots,&x_\nu,y_1,\dots,y_\nu\le h.\end{aligned}$$

This following simple statement relates $K_\nu(p,h,\mathbf{s})$ and $K_\nu(p,h,s_j)$, $j=1,\dots,\nu$.

Lemma 4. *We have*

$$K_\nu(p,h,\mathbf{s})\le\prod_{j=1}^{\nu}K_\nu(p,h,s_j)^{1/\nu}$$

Proof. Using the orthogonality of multiplicative characters, we write

$$\begin{aligned}K_\nu(p,h,\mathbf{s})&=\frac{1}{p-1}\sideset{}{^*}\sum_{1\le x_1,\dots,x_\nu,y_1,\dots,y_\nu\le h}\ \sum_{\chi}\chi\left(\frac{(x_1+s_1)\dots(x_\nu+s_\nu)}{(y_1+s_1)\dots(y_\nu+s_\nu)}\right)\\ &=\frac{1}{p-1}\sum_{\chi}\prod_{j=1}^{\nu}\sideset{}{^*}\sum_{1\le x_j,y_j\le h}\chi\left(\frac{x_j+s_j}{y_j+s_j}\right),\end{aligned}$$

where χ runs through all multiplicative characters modulo p and Σ^* indicates that summation does not involve $y_j \equiv -s_j \pmod p$. Using the Hölder inequality, we obtain the desired inequality. $\square$

4. Linear Congruences with Many Solutions

We need the following result, which in turn improves one of the results from [8].

Lemma 5. *Let $\gamma \in (0,1)$ and let I and J be two intervals containing h and H consecutive integers, respectively, and such that*

$$h \le H < \frac{\gamma p}{15}.$$

Assume that for some integer s the congruence

$$y \equiv sx \pmod p$$

has at least $\gamma h + 1$ solutions in $x \in I$, $y \in J$. Then there exist integers a and b with

$$|a| \le \frac{H}{\gamma h}, \qquad 0 < b \le \frac{1}{\gamma},$$

such that

$$s \equiv a/b \pmod p.$$

Proof. We can assume that $s \not\equiv 0 \pmod p$, as otherwise the statement is trivial. Making a shift of the set $I \times J$ by the solution (x_0, y_0) of our congruence with the least x_0 (here we use a natural ordering on I), without loss of generality we can assume that $I \subseteq [0, h]$, $J \subseteq [-H, H]$. Since $s \not\equiv 0 \pmod p$ and our congruence has a solution with $x \ne 0$, there exist integers a, b such that

$$s \equiv a/b \pmod p, \quad 0 < |a| \le H, \quad 0 < b \le h, \quad \gcd(a,b) = 1.$$

Thus, the equation

$$ax = by + pz$$

has at least $\gamma h + 1$ solutions in integer variables x, y, z with $x \in I$, $y \in J$. We have

$$|z| \le L,$$

where

$$L = \frac{|a|h + bH}{p}.$$

We consider two cases, $L < 1$ and $L \ge 1$.

Case 1: $L < 1$. Then $z = 0$ and we get that the equation $ax = by$ has at least $\gamma h + 1$ solutions in $x \in I, y \in J$. Since $\gcd(a,b) = 1$, we get that $x = bw$, $y = aw$ for some integer w and this should hold for

at least $\gamma h+1$ integers w (as there are at least $\gamma h+1$ pairs (x,y)). Therefore, $b\gamma h \le h$ and $|a|\gamma h \le H$ and the result follows.

Case 2: $L \ge 1$. Note that

$$L \le \frac{2hH}{p} \le \frac{2\gamma h}{15}.$$

Thus, by the pigeon-hole principle, there exists $z = z_0$ such that the equation $ax = by + pz_0$ has at least $\gamma h/(3L)$ solutions in variables $x \in I, y \in J$. We fix one such solution $(x_0, y_0) \in I \times J$ and get that the equality

$$a(x-x_0) = b(y-y_0),$$

holds for at least $\gamma h/(3L)$ pairs x, y with $|x-x_0| \le h, |y-y_0| \le H$. Since $\gcd(a,b)=1$, the equality implies

$$x - x_0 = bw, \quad y - y_0 = aw,$$

and this holds for at least $\gamma h/(3L)$ integers w. In particular, $|aw| \le H$ and $|bw| \le h$ for at least $\gamma h/(3L)$ integers w. Clearly, one of these integers w satisfies $|w| > \gamma h/(7L)$ and we therefore get

$$|a|\gamma h < 7LH, \quad b\gamma < 7L.$$

Together with the definition of L, this implies that

$$\gamma p = \frac{\gamma|a|h + \gamma b H}{L} < 14H,$$

contradicting the condition of our lemma. □

5. Congruences with Solution in Arbitrary Sets

We now use Lemma 5 to obtain a version of Theorem 19 below with $\nu = 2$, which applies to exponential congruences with variables from short intervals.

Lemma 6. *Let $\mathcal{X} \subseteq [1,h]$ be a set of integers with $h^3/(\#\mathcal{X}) < 0.002p$. Then for the number of solutions $L(p,\mathcal{X};s)$ of the congruence*

$$(x_1+s)(x_2+s) \equiv (y_1+s)(y_2+s) \not\equiv 0 \pmod p, \quad x_1,x_2,y_1,y_2 \in \mathcal{X} \tag{4}$$

we have

$$L(p,\mathcal{X};s) \le (\#\mathcal{X})^2 \exp\left(C \log h / \log\log h\right),$$

where C is an absolute constant.

Proof. Clearly, it is enough to estimate the contribution N to $L(p,\mathcal{X};s)$ of solutions of (4) with $x_i \ne y_j$, $1 \le i,j \le 2$.

Let $X = \#\mathcal{X}$. We also assume that

$$N > X^2 \exp\left(c_0 \frac{\log h}{\log\log h}\right)$$

for some large constant c_0 that is to be specified later. Observe that the last inequality implies

$$X > \exp\left(c_0 \frac{\log h}{\log\log h}\right)$$

due to the trivial estimate $N \le X^3$.

Note that for any Z we have

$$(x_1 + Z)(x_2 + Z) - (y_1 + Z)(y_2 + Z) = uZ - v,$$

where

$$u = x_1 + x_2 - y_1 - y_2, \quad v = y_1y_2 - x_1x_2.$$

By the pigeon-hole principle we have at least N/X solutions of (4) with the same $x_1 = x_1^*$. We claim that any pair (u, v) induced by these solutions occurs at most $\exp(c_0 \log h/\log\log h)$ times for some constant c_0. Indeed, fix a pair (u, v) and take $Z = -x_1^*$. We get

$$uZ - v = -(y_1 - x_1^*)(y_2 - x_1^*). \tag{5}$$

The number of solutions to (5) is bounded by $\exp(c_0 \log h/\log\log h)$. Each solution determines the numbers y_1, y_2 and the polynomial P, and for each y_1, y_2 we retrieve x_2. This proves the claim.

Therefore, there are at least $N \exp(-c_0 \log h/\log\log h)/X \ge X$ pairs (u, v) with

$$0 < |u| < 2h, \quad 0 < |v| < h^2,$$

such that

$$us \equiv v \pmod p.$$

We apply Lemma 5 (with $I = [-2h, 2h]$, $J = [-2h^2, 2h^2]$, $\gamma = X/(6h)$) and conclude that there are integers a and b satisfying conditions

$$|a| \le 6h^2/X, \quad 0 < |b| \le 6h/X, \quad s \equiv a/b \pmod p. \tag{6}$$

Now we multiply our original congruence

$$(x_1 + x_2 - y_1 - y_2)s + (x_1x_2 - y_1y_2) \equiv 0 \pmod p$$

by b and for $S = (x_1 + x_2 - y_1 - y_2)a + (x_1x_2 - y_1y_2)b$ we see that $S \equiv 0 \pmod p$. Since $h^3/X < 0.002p$, using (6) we derive that $|S| < p$. Thus, $S = 0$ and the congruence is converted to an equality, giving

$$(bx_1 + a)(bx_2 + a) = (by_1 + a)(by_2 + a),$$

and the result follows from the bound on the divisor function. □

Remark 7. *It is not difficult to show that the condition $h^3/(\#\mathcal{X}) < 0.002p$ of Lemma 6 can be relaxed to $h^3/(\#\mathcal{X}) < C_0p$, with any constant $C_0 > 0$.*

The following result is an extension of the well-known multiplicative energy estimate for pairs of intervals, frequently needed in character sum estimates, see [13, Theorem 3]. We use this result in the proof of Theorem 22 below.

Lemma 8. *Let A and B be positive integers with $AB \ll p$. Assume that I is an interval consisting on A consecutive integers, $\mathcal{Y}$ is a subset of an interval consisting on B consecutive integers with $0 \notin \mathcal{Y}$. Then the number of solutions of the congruence*

$$x_1y_1 \equiv x_2y_2 \pmod p, \quad (x_1, x_2, y_1, y_2) \in I \times I \times \mathcal{Y} \times \mathcal{Y}$$

is at most $(\#\mathcal{Y})^2 + A\#\mathcal{Y}p^{o(1)}$.

Proof. We can assume that $A < 0.1p$ and $B < 0.1p$, as otherwise the result becomes trivial.

Assume $I = \{\xi+1, \xi+2, \ldots, \xi+A\}$, $\mathcal{Y} \subseteq \{s+1, s+2, \ldots, s+B\}$. Let $\mathcal{Y}_0 = \mathcal{Y} - \{s\} \subseteq [1, B]$. We have to estimate the number of solutions of

$$(\xi + x_1)(s + y_1) \equiv (\xi + x_2)(s + y_2) \pmod p \tag{7}$$

with $1 \le x_1, x_2 \le A$, $y_1, y_2 \in \mathcal{Y}_0$. For a given pair y_1, y_2, the number of solutions of (7) with $1 \le x_1, x_2 \le A$ is clearly bounded by the number of solutions of the congruence

$$x_1(s + y_1) \equiv x_2(s + y_2) \pmod p$$

with $|x_1|, |x_2| \le A$. Thus, the number of solutions of the congruence (7) is bounded by the number of solutions of the congruence

$$x_1(s+y_1) \equiv x_2(s+y_2) \pmod p, \quad 1 \le |x_1|, |x_2| < A, \ y_1, y_2 \in \mathcal{Y}_0, \tag{8}$$

augmented by $(\#\mathcal{Y}_0)^2 = (\#\mathcal{Y})^2$.

Let N be the number of solutions of (8). We assume that $N \ge 2\#\mathcal{Y}$ since otherwise there is nothing to prove. For an appropriately fixed $y_1 \in \mathcal{Y}_0$ we obtain at least $N/\#\mathcal{Y} - 1 \ge N/(2\#\mathcal{Y})$ solutions with $y_2 \ne y_1$ (recall that $s + y_1 \not\equiv 0 \pmod p$ so $y_1 = y_2$ implies $x_1 = x_2$). If for each pair (u, v) of the form

$$(u, v) = (x_1 - x_2, x_2y_2 - x_1y_1) \tag{9}$$

we specify the polynomial

$$R_{u,v}(Z) = uZ - v = x_1(Z + y_1) - x_2(Z + y_2),$$

then we have

$$R_{u,v}(-y_1) \equiv x_2(y_1 - y_2) \pmod p.$$

Since $|x_2(y_1 - y_2)| \le 2AB \ll p$, we get at most $p^{o(1)}$ possibilities for x_2 and y_2 and hence for (x_1, x_2, y_2) (recall that y_1 is fixed and $y_1 \not\equiv y_2$

(mod p)). Thus, when (x_1, x_2, y_2) runs through the set of solutions, we get at least $Np^{o(1)}/\#\mathcal{Y}$ distinct polynomials $R_{u,v}(Z)$. Note that

$$R_{u,v}(s) = us - v \equiv 0 \pmod p$$

for each pair (u, v) of the form (9). Therefore, there are at least $Np^{o(1)}/\#\mathcal{Y}$ solutions $(u, v) \in \mathbb{Z} \times \mathbb{Z}$ of the congruence $us - v \equiv 0 \pmod p$ with $|u| \le 2A$, $|v| \le 2AB$. On the other hand, for any u there are $O(1)$ values of v satisfying $us - v \equiv 0 \pmod p$ since $AB \ll p$. Thus, $Np^{o(1)}/\#\mathcal{Y} \ll A$, and the desired result follows. ☐

The following result is used in estimating character sums with the divisor function.

Lemma 9. *For real X, Y and Z with*

$$X \ge 1, \qquad 2 \le Z \le Y, \qquad X^2YZ < p,$$

we consider the intervals $I = [1, X]$, $J = [1, Y]$ and denote by $\mathcal{P}$ the set of the primes $z \in (Z/2, Z]$. Then for $s \in \mathbb{F}_p^*$ the number of solutions of the congruence

$$\begin{gathered} x_2z_2(s + x_1y_1) \equiv x_1z_1(s + x_2y_2) \pmod p, \\ x_1, x_2 \in I,\ y_1, y_2 \in J,\ z_1, z_2 \in \mathcal{P} \end{gathered} \tag{10}$$

is at most $XYZp^{o(1)}$.

Proof. First we consider the case $s + x_1y_1 \equiv s + x_2y_2 \pmod p$. Then $x_1y_1 = x_2y_2$ since $1 \le x_1y_1, x_2y_2 \le XY < p$.

If, moreover, $s + x_1y_1 \equiv s + x_2y_2 \equiv 0 \pmod p$ then the number $x_1y_1 = x_2y_2$ is uniquely defined and so there are $p^{o(1)}$ possibilities for each of x_1, x_2, y_1, y_2. Hence, the number of such solutions is at most $Z^2p^{o(1)}$.

Assume now that

$$s + x_1y_1 \equiv s + x_2y_2 \not\equiv 0 \pmod p. \tag{11}$$

If also $x_1 = x_2$ then $y_1 = y_2$, $z_1 = z_2$, and we get $XY\#\mathcal{P} \le XYZ$ solutions. If $x_1 \ne x_2$, we specify the common value $u = x_1y_1 = x_2y_2$. Then x_1, x_2, y_1, y_2 are divisors of u and so there are $p^{o(1)}$ possibilities for each of them. For fixed x_1, x_2, y_1, y_2 the ratio z_1/z_2 is uniquely defined modulo p and $z_1/z_2 \not\equiv 1 \pmod p$ (since we have $x_1 \not\equiv x_2 \pmod p$ but $s+x_1y_1 \equiv s+x_2y_2 \pmod p$). Therefore z_1, z_2 are now uniquely defined too, as they are primes not exceeding $Z < \sqrt{p}$. Thus, the number of solutions to (10) satisfying (11) is at most $XYZ + XYp^{o(1)}$.

We now consider the case $s + x_1y_1 \not\equiv s + x_2y_2 \pmod p$. Let N be the number of solutions of (10) with $s + x_1y_1 \not\equiv s + x_2y_2 \pmod p$. In particular, this condition implies that $x_1z_1 \ne x_2z_2$ and $x_1y_1 \ne x_2y_2$. We

can assume that $N > 2XYZ$, as otherwise there is nothing to prove. There exist integers n_0, m_0 with $1 \le |n_0| < XZ, 1 \le m_0 \le Y$ such that we have at least $N/(2XYZ)$ solutions with $x_1z_1 - x_2z_2 = n_0, y_1 = m_0$. From (10) we see that

$$x_1x_2(y_1z_2 - y_2z_1) \equiv sn_0 \pmod p.$$

Thus, the number $x_1x_2(y_1z_2 - y_2z_1)$ is nonzero and well defined modulo p. Since its absolute value does not exceed $X^2YZ < p$, it may take at most two different integer values. Thus, we can retrieve x_1, x_2 and $y_1z_2 - y_2z_1$ with $p^{o(1)}$ possibilities. Once these numbers are retrieved, we use the equality

$$n_0m_0 + (y_1z_2 - y_2z_1)x_2 = z_1(x_1y_1 - x_2y_2)$$

and retrieve z_1 with $p^{o(1)}$ possibilities. Consequently, from $n_0 = x_1z_1 - x_2z_2$ we retrieve z_2, and then we retrieve y_2 from (10).

Thus, $N/(2XYZ) \le p^{o(1)}$ and the result follows. $\square$

6. Background on Algebraic Integers

Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}$ and let $\mathbb{Z}_\mathbb{K}$ be the ring of integers in $\mathbb{K}$. We recall that the logarithmic height of an algebraic number α is defined as the logarithmic height $H(P)$ of its minimal polynomial P, that is, the maximum logarithm of the largest (by absolute value) coefficient of P.

We need a bound of Chang [6, Proposition 2.5] on the divisor function in algebraic number fields.

Lemma 10. *Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}$ of degree $d = [\mathbb{K} : \mathbb{Q}]$. For any algebraic integer $\gamma \in \mathbb{Z}_K$ of logarithmic height at most $H \ge 2$, the number of pairs (γ_1, γ_2) of algebraic integers $\gamma_1, \gamma_2 \in \mathbb{Z}_K$ of logarithmic height at most H with $\gamma = \gamma_1\gamma_2$ is at most* $\exp(O(H/\log H))$, *where the implied constant depends on d.*

Now recall that the Mahler measure of a nonzero polynomial

$$P(Z) = a_dZ^d + \ldots + a_1Z + a_0 = a_d \prod_{j=1}^{d} (Z - \xi_j) \in \mathbb{C}[Z]$$

is defined as

$$M(P) = |a_d| \prod_{j=1}^{d} \max\{1, |\xi_j|\},$$

see [22, Chapter 3, Section 3]

We recall the following estimates, that follows immediately from a much more general [22, Theorem 4.4]:

Lemma 11. *For any nonzero polynomial P of degree d the following inequality holds*

$$2^{-d}e^{H(P)} \le M(P) \le (d+1)^{1/2}e^{H(P)}.$$

Corollary 12. *For any nonzero polynomials $Q_1, Q_2 \in \mathbb{C}[Z]$ we have*

$$H(Q_1Q_2) = H(Q_1) + H(Q_2) + O(1),$$

where the implied constant depends only on $\deg Q_1$ *and* $\deg Q_2$.

Lemma 13. *For any positive integer ν there is a constant C such that the following holds. If $P_1, P_2 \in \mathbb{Z}[Z]$, $P = P_1P_2$,*

$$P(Z) = \sum_{j=0}^{\nu} u_j Z^{\nu-j}$$

and for some $A > 0$ and $h > 0$ the coefficients of the polynomial P satisfy the inequalities

$$u_0 \neq 0, \qquad |u_j| \le Ah^j, \qquad j = 0, \dots, \nu,$$

then the polynomial P_1 has the form

$$P_1(Z) = \sum_{j=0}^{\mu} v_j Z^{\mu-j}$$

with

$$v_0 \neq 0, \quad |v_j| \le CAh^j \quad (j = 0, \dots, \mu).$$

Proof. We construct the polynomials

$$Q(Z) = P(hZ), \quad Q_1(Z) = P_1(hZ), \quad Q_2(Z) = P_2(hZ).$$

We have

$$e^{H(Q)} \le Ah^{\nu}.$$

Moreover,

$$e^{H(Q_2)} \ge h^{\nu-\mu}$$

since the leading coefficient of Q_2 is at least $h^{\nu-\mu}$. Therefore, by Corollary 12 we get

$$e^{H(Q_1)} \ll Ah^{\mu},$$

and the result follows. □

A particular case of Lemma 13 is the following statement (see, for example, [16, Theorem 6.32]).

Lemma 14. *Let $P, Q \in \mathbb{Z}[Z]$ be two univariate non-zero polynomials with $Q \mid P$. If P is of logarithmic height at most $H \ge 1$ then Q is of logarithmic height at most $H + O(1)$, where the implied constant depends only on* $\deg P$.

7. Background on Geometry of Numbers

Recall that a lattice in $\mathbb{R}^n$ is an additive subgroup of $\mathbb{R}^n$ generated by n linearly independent vectors. Take an arbitrary convex compact and symmetric with respect to 0 body $D \subseteq \mathbb{R}^n$. Recall that, for a lattice in $\Gamma \subseteq \mathbb{R}^n$ and $i = 1, \ldots, n$, the ith successive minimum $\lambda_i(D, \Gamma)$ of the set D with respect to the lattice Γ is defined as the minimal number λ such that the set λD contains i linearly independent vectors of the lattice Γ. Obviously, $\lambda_1(D, \Gamma) \le \ldots \le \lambda_n(D, \Gamma)$. We need the following result given in [2, Proposition 2.1] (see also [26, Exercise 3.5.6] for a simplified form that is still enough for our purposes).

Lemma 15. *We have,*

$$\#(D \cap \Gamma) \le \prod_{i=1}^{n} \left(\frac{2i}{\lambda_i(D, \Gamma)} + 1 \right).$$

Using an obvious inequality

$$\frac{2i}{\lambda_i(D, \Gamma)} + 1 \le (2i+1) \max \left\{ \frac{1}{\lambda_i(D, \Gamma)}, 1 \right\}$$

and denoting, as usual, by $(2n+1)!!$ the product of all odd positive numbers up to $2n+1$, we get the following

Corollary 16. *We have,*

$$\prod_{i=1}^{n} \min\{\lambda_i(D, \Gamma), 1\} \le (2n+1)!!(\#(D \cap \Gamma))^{-1}.$$

8. Common Solutions to Many Quadratic Congruences with Small Coefficients

We need the following statement, that can probably be extended in several directions.

Lemma 17. *For any positive integer $\nu \ge 3$ there are numbers $\eta > 0$ and $C > 0$, depending only on ν, such that if for a positive integer*

$$h \le \eta p^{1/\max\{\nu^2 - 2\nu - 2, \nu^2 - 3\nu + 4\}}$$

and $s \in \mathbb{F}_p$ there are $h^{\nu-1}$ different sequences $(A_1, \ldots, A_\nu) \in \mathbb{Z}^\nu$ with

$$|A_i| < 2^i h^i, \qquad i = 1, \ldots, \nu,$$

such that

$$A_1 s^{\nu-1} + \ldots + A_{\nu-1} s + A_\nu \equiv 0 \pmod p,$$

then we have the following:

(i) *If* $\nu = 3$, *then*
$$s \equiv a/b \pmod p$$
for some integers a, b *with* $a \ll h^{3/2}$, $b \ll h^{1/2}$.

(ii) *If* $\nu = 4$ *then there is a nonzero sequence* $(B_2, B_3, B_4) \in \mathbb{Z}^3$ *with*
$$|B_i| < Ch^{i-2}, \qquad i = 2, 3, 4,$$
and such that
$$B_2 s^2 + B_3 s + B_4 \equiv 0 \pmod p.$$

(iii) *If* $\nu \geq 5$ *then there is a nonzero sequence* $(B_3, \ldots, B_\nu) \in \mathbb{Z}^{\nu-2}$ *with*
$$|B_i| < Ch^{i-2-1/(\nu-2)}, \qquad i = 3, \ldots, \nu,$$
and such that
$$B_3 s^{\nu-3} + \ldots + B_{\nu-1} s + B_\nu \equiv 0 \pmod p.$$

Proof. We can assume that $h \geq h_0(\nu)$ for some appropriate constant $h_0(\nu)$, depending only on ν. We define the lattice
$$\Gamma = \{(u_1, \ldots, u_\nu) \in \mathbb{Z}^\nu \ : \ u_1 s^{\nu-1} + \ldots + u_{\nu-1} s + u_\nu \equiv 0 \pmod p\}$$
and the body
$$D = \{(u_1, \ldots, u_\nu) \in \mathbb{Z}^\nu \ : \ |u_1| < 2^\nu h, \ldots, |u_{\nu-1}| < 2^\nu h^{\nu-1}, \ |u_\nu| < 2^\nu h^\nu\}.$$
We know that
$$\#(D \cap \Gamma) \geq h^{\nu-1}.$$
Therefore, by Corollary 16, the successive minima $\lambda_i = \lambda_i(D, \Gamma)$, $i = 1, \ldots, \nu$, satisfy the inequality
$$\prod_{i=1}^{\nu} \min\{1, \lambda_i\} \ll h^{1-\nu}. \tag{12}$$
In particular, $\lambda_1 \leq 1$.

We consider separately the following seven cases.

Case 1: $\lambda_\nu \leq 1$. By definition of λ_i, there are linearly independent vectors $(u_1^i, \ldots, u_\nu^i) \in \lambda_i D \cap \Gamma$, $i = 1, \ldots, \nu$. By (12), we have $\lambda_1 \ldots \lambda_\nu \ll h^{1-\nu}$. We consider the determinant
$$\Delta = \det \begin{pmatrix} u_1^1 & \ldots & u_\nu^1 \\ \ldots & \ldots & \ldots \\ u_1^\nu & \ldots & u_\nu^\nu \end{pmatrix}.$$
Clearly,
$$\Delta \ll h^{(\nu^2+\nu)/2} \lambda_1 \ldots \lambda_\nu \ll h^{(\nu^2-\nu+2)/2}.$$

On the other hand, from

$$u_1^i s^{\nu-1} + \ldots + u_\nu^i \equiv 0 \pmod p, \qquad i = 1, \ldots, \nu$$

we conclude that Δ is divisible by p. Therefore, for a sufficiently large $h_0(\nu)$ we derive $\Delta = 0$, but this contradicts linear independence of the vectors $(u_1^i, \ldots, u_\mu^i)$, $i = 1, \ldots, \nu$. Thus this case is impossible.

Case 2: $\lambda_{\nu-1} \le 1$, $\lambda_\nu > 1$. We can assume that $s \not\equiv 0 \pmod p$. By definition, there are linearly independent vectors $(u_1^i, \ldots, u_\nu^i) \in \lambda_i D \cap \Gamma$, $i = 1, \ldots, \nu-1$. By (12), we have

$$\lambda_1 \ldots \lambda_{\nu-1} \ll h^{1-\nu}. \tag{13}$$

Again by definition,

$$\begin{cases} u_1^1 s^{\nu-1} + \ldots + u_{\nu-1}^1 s & \equiv & -u_\nu^1 \pmod p; \\ & \ldots & \\ u_1^{\nu-1} s^{\nu-1} + \ldots + u_{\nu-1}^{\nu-1} s & \equiv & -u_\nu^{\nu-1} \pmod p. \end{cases} \tag{14}$$

Let

$$\Delta_0 = \det \begin{pmatrix} u_1^1 & \ldots & u_{\nu-1}^1 \\ \ldots & \ldots & \ldots \\ u_1^{\nu-1} & \ldots & u_{\nu-1}^{\nu-1} \end{pmatrix}$$

and for $i = 1, 2, \ldots, \nu-1$ let $\Delta_{\nu-i}$ be the determinant of the matrix obtained from the matrix

$$\begin{pmatrix} u_1^1 & \ldots & u_{\nu-1}^1 \\ \ldots & \ldots & \ldots \\ u_1^{\nu-1} & \ldots & u_{\nu-1}^{\nu-1} \end{pmatrix}$$

by replacing its i-th column by $(-u_\nu^1, -u_\nu^2, \ldots, -u_\nu^{\nu-1})$. From (13) we conclude that

$$\Delta_j \ll h^{(\nu-2)(\nu-1)/2+j}, \qquad j = 0, 1, \ldots, \nu-1. \tag{15}$$

If $\Delta_0 \equiv 0 \pmod p$, then $\Delta_0 = 0$ and from the system of congruences (14) we derive that $\Delta_j \equiv 0 \pmod p$ and thus, in view of (15), we have $\Delta_j = 0$ for all $1 \le j \le \nu-1$. Hence, the rank of the matrix

$$\begin{pmatrix} u_1^1 & \ldots & u_{\nu-1}^1 & u_\nu^1 \\ \ldots & \ldots & \ldots & \ldots \\ u_1^{\nu-1} & \ldots & u_{\nu-1}^{\nu-1} & u_\nu^{\nu-1} \end{pmatrix}$$

is strictly less than $\nu-1$, which contradicts to the linear independence of the corresponding vectors.

Thus, we have that

$$\Delta_0 \not\equiv 0 \pmod p.$$

Next, from the system (14) we find that

$$s^j \equiv \Delta_j/\Delta_0 \pmod p, \qquad j = 1, \ldots, \nu-1. \tag{16}$$

Since $s \not\equiv 0 \pmod p$, $\Delta_j \not\equiv 0 \pmod p$. Comparing s and s^2 we obtain

$$\Delta_1^2 \equiv \Delta_0\Delta_2 \pmod p.$$

Since both hand sides are $O\left(h^{\nu^2-3\nu+4}\right)$, we see that

$$\Delta_1^2 = \Delta_0\Delta_2.$$

Thus, there exist coprime integers integers a, b such that for $r_2 = \gcd(\Delta_0, \Delta_2)$ we have

$$\Delta_0 = r_2 b^2, \quad \Delta_2 = r_2 a^2 \quad \Delta_1 = r_2 ab. \tag{17}$$

Now we claim that there are integers $r_2, \ldots, r_{\nu-1}$ such that the equalities

$$\Delta_0 = r_j b^j, \quad \Delta_1 = r_j a b^{j-1}, \quad \Delta_j = r_j a^j \tag{18}$$

hold for all $j = 2, 3, \ldots, \nu-1$.

We prove this claim by induction on j. For $j = 2$ the statement follows from (17). We now assume that (18) holds for some $2 \le j \le \nu - 2$ and prove it with j replaced by $j+1$.

Comparing s and s^{j+1} in (16), we get

$$\Delta_1^{j+1} \equiv \Delta_{j+1}\Delta_0^j \pmod p.$$

We substitute here Δ_0 and Δ_1 in accordance to our induction hypothesis. After cancellations (recall that $\Delta_j \not\equiv 0 \pmod p$), we get

$$r_j a^{j+1} \equiv \Delta_{j+1} b \pmod p. \tag{19}$$

In view of the induction hypothesis, the left hand side of (19) is of size at most

$$|r_j a^j|^{(j+1)/j} \ll |\Delta_j|^{(j+1)/j} \ll h^{((\nu-2)(\nu-1)/2+j)(j+1)/j}.$$

Since

$$((\nu-2)(\nu-1)/2+j)(j+1)/j \le ((\nu-2)(\nu-1)/2+\nu-2)3/2 < \nu^2-3\nu+4,$$

we get

$$|r_j a^j|^{(j+1)/j} \ll h^{\nu^2-3\nu+4}.$$

Thus, we see that the left hand side of (19) is less than $p/2$. Again in view of the induction hypothesis we have $|b| \le |\Delta_0|^{1/j}$. Hence, in view of (15), the right hand side of (19) is

$$\Delta_{j+1} b \ll h^{(\nu-2)(\nu-1)/2+j+1} h^{(\nu-2)(\nu-1)/(2j)} \ll h^{\nu^2-3\nu+4}.$$

Thus, again from the condition of the lemma we get that the right hand side is less, than $p/2$. Hence, the congruence is converted to the equality

$$r_j a^{j+1} = \Delta_{j+1} b.$$

Since $\gcd(a, b) = 1$, this implies that for some integer r_{j+1} we have

$$\Delta_{j+1} = r_{j+1} a^{j+1}, \quad r_j = b r_{j+1}.$$

Replacing r_j with its value given by the induction hypothesis, we arrive to (18).

We have

$$s \equiv \Delta_1/\Delta_0 \equiv a/b \pmod p.$$

In our intermediate statement we take $j = \nu - 1$ and

$$|a| \le |\Delta_{\nu-1}|^{1/(\nu-1)} \ll h^{\nu/2}, \quad |b| \le |\Delta_0|^{1/(\nu-1)} \ll h^{\nu/2-1}.$$

If $\nu = 3$ then we are done. If $\nu = 4$, then the statement follows by taking $B_2 = 0, B_3 = -a, B_4 = b$. If $\nu \ge 5$, then the statement follows by taking

$$B_\nu = -a, \quad B_{\nu-1} = b$$

and $B_j = 0$ for $j < \nu - 1$.

Below we use the following argument. As in Cases 1 and 2, if $\lambda_r \le 1$ for some $r = 1, \ldots, \nu$, then, by definition there are linearly independent vectors $(u_1^j, \ldots, u_\nu^j) \in \lambda_j D \cap \Gamma$, $j = 1, \ldots, r$. Clearly, we can assume that $\gcd(u_1^j, \ldots, u_\nu^j) = 1$. Next, we construct linear independent polynomials

$$P_j(Z) = \sum_{i=1}^{\nu} u_i^j Z^{\nu-i}, \quad j = 1, \ldots, r.$$

We note that $P_j(s) \equiv 0 \pmod p$ for $j = 1, \ldots, r$.

Case 3: $\lambda_1 \le 31 h^{-2}$ for $\nu \le 4$ and $\lambda_1 \le h^{-2-1/(\nu-2)}$ for $\nu \ge 5$. For $\nu \le 4$ we have $u_i^1 \ll h^{i-2}$. Therefore, $u_1^1 = 0$ provided that $h_0(\nu)$ is large enough. If $\nu = 3$ we take $a = -u_3^1$, $b = u_2^1$. If $\nu \ge 4$ we take $B_i = u_i^1$ where $2 \le i \le \nu$ for $\nu = 4$ and $3 \le i \le \nu$ for $\nu \ge 5$.

We observe that if $\nu = 3$ and $\lambda_2 \ge 1$ then, by Corollary 16 we get $\lambda_1 \le 15 h^{-2}$. Thus, for $\nu = 3$ at least one of Cases 1–3 holds and the proof is complete. Throughout the following we always assume that $\nu \ge 4$.

If Case 3 does not hold, then we have $\lambda_2 \le 1$ by (12). Thus, a polynomial P_2 is well-defined.

We denote $R_j = \gcd(P_1, P_j)$ if $\mathrm{Res}(P_1, P_j) = 0$ for some $j > 1$. We have $R_j(s) \equiv 0 \pmod p$. If $R_j \ne \pm P_1$ then $\deg R_j \le \deg P_1 - 1$ (taking

into account that the coefficients of P_1 are coprime). If, moreover, $\lambda_{\nu-1} > 1$ then, by (12), we have

$$\lambda_1 \ll h^{-1-1/(\nu-2)}. \tag{20}$$

This inequality implies $u_1^1 = 0$, that is, $\deg P_1 \le \nu - 2$ (provided that $h_0(\nu)$ is large enough) and $u_i^1 \ll h^{i-1-1/(\nu-2)}$ for $i \ge 2$. Therefore, if $R_j \ne \pm P_1$ and $\lambda_{\nu-1} > 1$, then, by Lemma 13, the coefficients of the polynomial R_j satisfy the statement of the theorem. Hence, we can suppose that P_1 divides P_j.

Case 4: $\lambda_1 > 31h^{-2}$, $\lambda_2 \le 31h^{-1}$ for $\nu = 4$ and $\lambda_1 > h^{-2-1/(\nu-2)}$, $\lambda_2 \le h^{-1-1/(\nu-2)}$ for $\nu \ge 5$; $\lambda_{\nu-1} > 1$.

Suppose that $\nu = 4$. We take $M = 3$, $N = 4$, $\sigma = \log(16\lambda_1)/\log h+2$, $\vartheta = \log(16\lambda_2)/\log h + 1$. Condition (i) of Lemma 2 holds, and we can use Corollary 3 taking into account that we have, by (12), $\lambda_1\lambda_2 \ll h^{-3}$. Hence, $\mathrm{Res}(P_1, P_2) \ll h^8$ and $|\mathrm{Res}(P_1, P_2)| < p$ provided that η has been chosen small enough. On the other hand, since $P_1(s) \equiv P_2(s) \equiv 0 \pmod p$, $\mathrm{Res}(P_1, P_2)$ is divisible by p. Consequently, $\mathrm{Res}(P_1, P_2) = 0$. By our supposition P_1 divides P_2. Since P_1 and P_2 are linearly independent, we conclude that $\deg P_1 \le \deg P_2 - 1$. Using the inequality $\lambda_2 \le 31h^{-1}$ and Lemma 13, we see that the coefficients of the polynomial R_j satisfy the statement of the theorem.

For $\nu \ge 5$ the proof is similar. The inequality $\lambda_2 \le h^{-1-1/(\nu-2)}$ implies $u_1^2 = 0$ provided that $h_0(\nu)$ is large enough. So, $\deg P_1 \le \nu-2$, $\deg P_2 \le \nu-2$. Now we take $M = N = \nu-1$, $\sigma = \log(2^\nu\lambda_1)/\log h+2$, $\vartheta = \log(2^\nu\lambda_2)/\log h+2$. The condition (iii) of Lemma 2 holds, and we can use Corollary 3. By (12), we get $\lambda_1\lambda_2 \ll h^{-2-2/(\nu-2)}$. Therefore, gives $\mathrm{Res}(P_1, P_2) \ll h^{\nu^2-2\nu-2}$. The rest is essentially the same as for $\nu = 4$.

Now suppose that $\nu = 4$ and neither of Cases 1–4 holds. We conclude that

$$\prod_{i=1}^{4} \min\{\lambda_i(D, \Gamma), 1\} > 31^2h^{-3} \ge 31^2(\#(D \cap \Gamma))^{-1}.$$

However, this inequality contradicts Corollary 16. Thus, for $\nu = 4$ at least one of Cases 1–4 holds and the proof is complete. Throughout the following we always assume that $\nu \ge 5$.

In the rest of the proof we estimate $\mathrm{Res}(P_1, P_j)$ for $j = 2$ or $j = 3$ by Corollary 3 considering that $\lambda_1 > h^{-2-1/(\nu-2)}$ and $\lambda_2 > h^{-1-1/(\nu-2)}$. We take $M = \nu-1$, $\sigma = \log(2^\nu\lambda_1)/\log h+2$. If we know that $\deg P_j \le \nu-2$ then we take $N = \nu - 1$, $\vartheta = \log(2^\nu\lambda_1)/\log h + 2$. The inequality $\lambda_j > h^{-1-1/(\nu-2)}$ implies $\vartheta \ge 0$, and condition (ii) of Lemma 2 holds.

Otherwise, we take $N = \nu$, $\vartheta = \log(2^\nu\lambda_1)/\log h + 1$ and have the condition (iii) of Lemma 2.

Case 5: $\nu = 5$, $\lambda_1 > h^{-2-1/3}$, $\lambda_2 > h^{-1-1/3}$, $\lambda_4 > 1$. Assuming that $\lambda_3 > 1$ we get contradiction with inequality (12) provided that $h_0(5)$ is large enough. Hence, $\lambda_3 \le 1$. Using again (12) we get

$$\lambda_1 \ll h^{-4/3}, \quad \lambda_1\lambda_j \le \lambda_1\lambda_3 \ll h^{-8/3} \quad (j = 2, 3).$$

Now we are in position to apply Corollary 3 to the polynomials P_1, P_j (recalling that $\deg P_1 \le 3$, $\deg P_j \le 4$). We get

$$\mathrm{Res}(P_1, P_j) \ll h^{23}(\lambda_1\lambda_j)^3\lambda_1 \ll h^{41/3}.$$

Hence, $|\mathrm{Res}(P_1, P_2)| < p$ provided that $h_0(5)$ has been chosen large enough. As before, we deduce that $\mathrm{Res}(P_1, P_j) = 0$. By our supposition, P_1 divides P_2 and P_3. Since P_1, P_2 and P_3 are linearly independent, we conclude that $\deg P_1 \le \deg P_j - 2$ for $j = 2$ or $j = 3$. Using the inequality $\lambda_j \ll h^{-1/3}$ and Lemma 13, we see that P_1 has the form $AZ^2 + BZ + C$ where $A \ll h^{2/3}$, $B \ll h^{5/3}$, $C \ll h^{8/3}$ as required.

Now the proof is complete for $\nu = 5$.

Case 6: $\nu \ge 6$, $\lambda_3 \le h^{-1}2^{-\nu}$, $\lambda_{\nu-1} > 1$. Since $\lambda_j \le h^{-1}2^{-\nu}$ for $j = 1, 2, 3$, we have $u_1^j = 0$ and $\deg P_j \le \nu - 2$. We conclude from (12) that $\lambda_1^2\lambda_3^{\nu-4} \ll h^{1-\nu}$. Hence,

$$\lambda_1\lambda_j \le \lambda_1\lambda_3 \ll h^{-2-2/(\nu-2)}$$

for $j = 2, 3$. By Corollary 3 we have $\mathrm{Res}(P_1, P_j) \ll h^{\nu^2-2\nu-2}$. As in the previous case, we deduce that $\deg P_1 \le \deg P_j - 2$ for $j = 2$ or $j = 3$. Using the inequality $\lambda_j \ll h^{-1}$ and Lemma 13, we conclude that $\deg P_1 \le \nu - 4$ and $u_j^1 \ll h^{j-3}$ for $j \ge 4$, and the desired result follows.

Case 7: $\nu \ge 6$, $\lambda_1 > h^{-2-1/(\nu-2)}$, $\lambda_2 > h^{-1-1/(\nu-2)}$, $\lambda_3 > h^{-1}2^{-\nu}$, $\lambda_{\nu-1} > 1$. Taking into account lower bounds for λ_2 and $\lambda_j\,(j = 4, \dots, \nu - 2)$ we conclude from (12) that

$$\lambda_1\lambda_j \le \lambda_1\lambda_3 \ll h^{-3+1/(\nu-2)}$$

for $j = 2, 3$. Next, using the lower bound for λ_1, we get

$$\lambda_3 \ll h^{-(\nu-4)/(\nu-2)}. \tag{21}$$

Assuming that $h_0(\nu)$ is large enough, we have $\lambda_3 \le 1$. Hence, the polynomials P_j are defined for $j \le 3$; moreover, $\deg P_1 \le \nu - 2$ and $\deg P_j \le \nu - 1$ for $j = 2, 3$. By Corollary 3, we have for $j = 2, 3$

$$\mathrm{Res}(P_1, P_j) \ll h^{\nu^2-2}(\lambda_1\lambda_3)^{\nu-1}\lambda_3^{-1} \ll h^u,$$

where

$$u = \nu^2 - 2 - (3 - 1/(\nu-2))(\nu-1) + 1 = \nu^2 - 3\nu + 3 + 1/(\nu-2) < \nu^2 - 3\nu + 4.$$

As in the previous cases, we consequently conclude that $\mathrm{Res}(P_1, P_j) = 0$, P_1 divides P_j for $j = 2, 3$ and $\deg P_1 \le \deg P_j - 2$ for $j = 2$ or $j = 3$. Using (21) and Lemma 13 completes the proof. □

Remark 18. *One can try to separate the case*

$$\lambda_{\nu-2} \le h^{-1} 2^{-\nu}, \quad \lambda_{\nu-1} > 1$$

and to use the same arguments as in Case 2. We expect that by this way it is possible to improve slightly the exponent in the restriction on h for $\nu \ge 7$, however we have not attempted to do so.

9. Product Sets in $\mathbb{F}_p$

Here we obtain some upper bounds on $K_\nu(p, h, s)$ that hold for all primes.

Theorem 19. *Let $\nu \ge 3$ be a fixed integer and let*

$$e_\nu = \max\{\nu^2 - 2\nu - 2, \nu^2 - 3\nu + 4\}.$$

Then we have the bound

$$K_\nu(p, h, s) \le \left(\frac{h^\nu}{p^{\nu/e_\nu}} + 1 \right) h^\nu \exp\left(c(\nu) \frac{\log h}{\log\log h} \right),$$

where $c(\nu)$ depends only on ν.

Proof. Using Lemma 4, we see that we can assume that

$$h < \eta p^{1/e_\nu} \tag{22}$$

for some small constant $\eta > 0$.

It is more convenient to include the case $\nu = 2$. For $\nu = 2$ we know from [4] the bound

$$K_\nu(p, h, s) \le h^2 \exp\left(c \frac{\log h}{\log\log h} \right), \qquad h \le p^{1/3}, \tag{23}$$

where $c > 0$ (see also Lemma 6). For $\nu \ge 3$ we prove by induction on ν the estimate

$$K_\nu(p, h, s) \le h^\nu \exp\left(c(\nu) \frac{\log h}{\log\log h} \right), \qquad h \le \eta_\nu p^{1/e_\nu}. \tag{24}$$

By the induction hypothesis (the inequalities (23) for $\nu = 3$ and (24) for $\nu > 3$), the set $(x_1, \ldots, x_\nu)$ of solutions of the congruence (3) for which $x_i = y_j$ for some i, j contributes to $K_\nu(p, h, s)$ at most

$$h^\nu \nu^2 \exp\left(c(\nu-1)\frac{\log h}{\log\log h}\right) \le h^\nu \exp\left(0.5c(\nu)\frac{\log h}{\log\log h}\right), \tag{25}$$

provided that h is large enough and we also choose $c(\nu) > 2c(\nu - 1)$.

We associate with any solution of (3) such that

$$\{x_1, \ldots, x_\nu\} \cap \{y_1, \ldots, y_\nu\} = \emptyset, \tag{26}$$

the polynomials

$$P(Z) = (x_1 + Z)\ldots(x_\nu + Z), \qquad Q(Z) = (y_1 + Z)\ldots(y_\nu + Z),$$

and

$$R(Z) = P(Z) - Q(Z). \tag{27}$$

We note that each such polynomial $R(Z)$ is nonzero and has a form

$$R(Z) = A_1 Z^{\nu-1} + \ldots + A_{\nu-1}Z + A_\nu \in \mathbb{Z}[Z],$$

with $|A_i| \le 2^\nu h^i$, $i = 1, \ldots, \nu$. In particular, since $R(s) \equiv 0 \pmod p$, it follows that $R(Z)$ is not a constant polynomial.

Let N be the number of the solutions of (3) satisfying (26). We proceed as in the proof of Lemma 6. By the pigeon-hole principle we have at least N/h solutions with the same $x_1 = x_1^*$. We claim that any polynomial R induced by these solutions occurs at most $\exp(c_0(\nu)\log h/\log\log h)$ times for some constant $c_0(\nu)$ depending only on ν. Indeed, fix R and take $Z = -x_1^*$. We get

$$M = -Q(-x_1^*) = z_1 \ldots z_\nu, \tag{28}$$

where $M = R(-x_1^*)$, $z_i = -x_1^* + y_i$, $i = 1, \ldots, \nu$. The number of solutions to (28) is bounded by $\exp(c_0(\nu)\log h/\log\log h)$. Each solution determines the numbers $y_1, \ldots, y_\nu$ and the polynomial P, and for each P there are at most $(\nu - 1)!$ solutions of (3). This proves the claim.

Therefore, we can take

$$N_1 \ge \exp\left(-c_0(\nu)\frac{\log h}{\log\log h}\right) h^{-1} N \tag{29}$$

solutions of (3) satisfying (26) with $x_1 = x_1^*$ and distinct polynomials R. Assume that

$$N_1 \ge h^{\nu-1}.$$

as otherwise there is nothing to prove (if we take $c(\nu) > c_0(\nu) + 1$). Now we are in position to use Lemma 17.

If $\nu = 3$, then we have

$$s \equiv a/b \pmod p$$

for some integers a, b with $a \ll h^{3/2}$, $b \ll h^{1/2}$.

Note that for each solution $(x_1, x_2, x_3, y_1, y_2, y_3)$ that contributes to N, we have

$$\begin{aligned} 0 &\equiv (x_1+s)(x_2+s)(x_3+s) - (y_1+s)(y_2+s)(y_3+s) \\ &\equiv (x_1+x_2+x_3-y_1-y_2-y_3)s^2 \\ &\qquad +(x_1x_2+x_2x_3+x_3x_1-y_1y_2-y_2y_3-y_3y_1)s \\ &\qquad\quad +(x_1x_2x_3-y_1y_2y_3) \pmod p. \end{aligned}$$

Recalling that $s \equiv ab^{-1} \pmod p$, we now obtain

$$\begin{aligned} &(x_1+x_2+x_3-y_1-y_2-y_3)a^2 \\ &\qquad +(x_1x_2+x_2x_3+x_3x_1-y_1y_2-y_2y_3-y_3y_1)ab \\ &\qquad\quad +(x_1x_2x_3-y_1y_2y_3)b^2 \equiv 0 \pmod p. \end{aligned} \tag{30}$$

Since the right hand side of (30) is $\ll h^4$ we obtain the equation

$$(bx_1+a)(bx_2+a)(bx_3+a) = (by_1+a)(by_2+a)(by_3+a) + \lambda bp,$$

where

$$1 \le x_i, y_i \le h, \qquad i = 1, 2, 3,$$

with some $\lambda \ll h^4/p + 1 \ll 1$. Recalling the well-known bound on the divisor function (a special case of Lemma 10) we obtain the result.

If $\nu \ge 4$, then, by Lemma 17 we get a polynomial

$$R^*(Z) = B_2 Z^{\nu-2} + \ldots + B_{\nu-1} Z + B_\nu$$

with $R^*(s) \equiv 0 \pmod p$,

$$|B_i| < 2^\nu h^{i-2}, \qquad i = 2, 3, 4,$$

for $\nu = 4$ and

$$B_2 = 0, \quad |B_i| < 2^\nu h^{i-2-1/(\nu-2)}, \qquad i = 3, \ldots, \nu,$$

for $\nu \ge 5$.

We fix such a polynomial R^* and consider an arbitrary solution of (3), satisfying (26) with $x_1 = x_1^*$ and take the corresponding polynomial R given by (27). Using Corollary 3, and recalling the assumption (22), we see that

$$\mathrm{Res}(R, R^*) \ll p^8$$

for $\nu = 4$ and

$$\mathrm{Res}(R, R^*) \ll p^{\nu^2-2\nu-2-1/(\nu-2)}$$

for $\nu \geq 5$. Thus,

$$|\mathrm{Res}(R, R^*)| < p, \tag{31}$$

provided that $\eta > 0$ is small enough. Since

$$R(s) \equiv R^*(s) \equiv 0 \pmod p$$

we also have

$$\mathrm{Res}(R, R^*) \equiv 0 \pmod p. \tag{32}$$

Therefore, we see from (31) and (32) that

$$\mathrm{Res}(R, R^*) = 0.$$

Hence, every polynomial R has a common root with R^*. Thus, by Lemma 14, we find an algebraic number β of logarithmic height $O(\log h)$ in an extension $\mathbb{K}$ of $\mathbb{Q}$ of degree $[\mathbb{K} : \mathbb{Q}] \leq \nu$ such that the equation

$$(x_1 + \beta) \ldots (x_\nu + \beta) = (y_1 + \beta) \ldots (y_\nu + \beta) \neq 0, \tag{33}$$

where

$$1 \leq x_i, y_i \leq h \qquad \text{and} \qquad x_1 = x_1^* \neq y_i, \quad i = 1, \ldots, \nu,$$

has at least N_1/ν solutions. Now we have that

$$\beta = \frac{\alpha}{q},$$

where α is an algebraic integer of height at most $O(\log h)$ and q is a positive integer $q \ll h^\nu$, see [23]. From the basic properties of algebraic numbers it now follows that the numbers

$$qx_i + \alpha \qquad \text{and} \qquad qy_i + \alpha, \qquad i = 1, \ldots, \nu,$$

are algebraic integers of $\mathbb{K}$ of height at most $O(\log h)$.

Therefore, we conclude that for a sufficiently large h the equation (33) has at most

$$\exp\left(C(\nu)\frac{\log h}{\log\log h}\right) \leq \exp\left(0.5c(\nu)\frac{\log h}{\log\log h}\right) \tag{34}$$

solutions, where $C(\nu)$ is the implied constant of Lemma 10 and we also assume that $c(\nu) > 2C(\nu)$. This implies the bound (24) and completes the proof. $\square$

For a set $\mathcal{A} \subseteq \mathbb{F}_p$ we denote

$$\mathcal{A}^{(\nu)} = \{a_1 \ldots a_\nu \ : \ a_1, \ldots, a_\nu \in \mathcal{A}\}.$$

Corollary 20. *Let $\nu \ge 3$ be a fixed integer and let*

$$e_\nu = \max\{\nu^2 - 2\nu - 2, \nu^2 - 3\nu + 4\}.$$

Assume that for some sufficiently large positive integer h and prime p we have

$$h < p^{1/e_\nu}.$$

For $s \in \mathbb{F}_p$ we consider the set

$$\mathcal{A} = \{x + s \ : \ 1 \le x \le h\} \subseteq \mathbb{F}_p.$$

Then

$$\#\mathcal{A}^{(\nu)} > \exp\left(-c(\nu)\frac{\log h}{\log\log h}\right) h^\nu,$$

where $c(\nu)$ depends only on ν.

10. Points on Exponential Curves

This result improves the bound of [9, Corollary 3].

Theorem 21. *Let g be of multiplicative order t modulo p and let $\gcd(a,p) = 1$. Let I_1 and I_2 be two intervals consisting on h_1 and h_2 consecutive integers respectively, where $h_2 \le t$. Then the number $R_{a,g,p}(I_1, I_2)$ of solutions of the congruence*

$$x \equiv ag^z \pmod p, \quad (x, z) \in I_1 \times I_2$$

is bounded by

$$R_{a,g,p}(I_1, I_2) < \left(h_1 p^{-2/5} + 1\right) h_2^{1/2+o(1)}.$$

Proof. By the pigeonhole principle, there exists an interval $I_{11} \subseteq I_1$ of length

$$|I_{11}| = \min\{h_1, \lfloor p^{2/5} \rfloor\}$$

such that

$$R_{a,g,p}(I_1, I_2) \le \left(\frac{2h_1}{p^{2/5}} + 1\right) R_{a,g,p}(I_{11}, I_2),$$

where $R_{a,g,p}(I_{11}, I_2)$ is the number of solutions of the congruence

$$x \equiv ag^z \pmod p, \quad (x, z) \in I_{11} \times I_2. \tag{35}$$

It is enough to prove that for any fixed $\varepsilon > 0$ we have

$$R_{a,g,p}(I_{11}, I_2) < h_2^{1/2+\varepsilon}.$$

Let $\mathcal{X} \subseteq I_{11}$ be the set of x for which the congruence (35) is satisfied for some $z \in I_2$. Let

$$T(\lambda) = \#\{\lambda \in \mathbb{F}_p^* \ : \ \lambda \equiv x_1 x_2 \pmod p \text{ for some } x_1, x_2 \in \mathcal{X}\}.$$

Then obviously,

$$\#\{\lambda \ : \ T(\lambda) > 0\} \le 2h_2.$$

Hence, using the Cauchy inequality, we obtain

$$\begin{aligned}\#\{x_1x_2 \equiv & y_1y_2 \pmod p \ : \ x_1, x_2, y_1, y_2 \in \mathcal{X}\} \\ &= \sum_{\lambda \in \mathbb{F}_p^*} T(\lambda)^2 \ge (2h_2)^{-1}\left(\sum_{\lambda\in\mathbb{F}_p^*} T(\lambda)\right)^2 = \frac{(\#\mathcal{X})^4}{2h_2}.\end{aligned} \tag{36}$$

Assume that $\#\mathcal{X} > h_2^{1/2+\delta}$ for some $\delta > 0$ (otherwise there is nothing to prove). In this case

$$\frac{|I_{11}|^3}{\#\mathcal{X}} < \frac{\min\{h_2^3, p^{6/5}\}}{h_2^{1/2+\delta}} = \min\{h_2^{5/2-\delta}, p^{6/5}h_2^{-1/2-\delta}\} = o(p).$$

So Lemma 6 applies and implies that

$$\#\{x_1x_2 \equiv y_1y_2 \pmod p \ : \ x_1, x_2, y_1, y_2 \in \mathcal{X}\} \ll (\#\mathcal{X})^2 h_2^{o(1)}. \tag{37}$$

Clearly (36) and (37) contradict the assumption $\#\mathcal{X} > h_2^{1/2+\delta}$ and the result follows. $\square$

In particular, Theorem 21 extends the range $h \le p^{1/3}$ given in [9] up to $h \le p^{2/5}$ under which the bound $R_{a,g,p}(I_1, I_2) = h^{1/2+o(1)}$ holds for $h = h_1 = h_2$.

11. Double Character Sums Estimates

We first point out the following improvement of the result from Friedlander and Iwaniec [13, Theorem 3].

Theorem 22. *Let $AB < p$, $B \le A$, $\mathcal{A} \subseteq [M, M + A]$ and $\mathcal{B} \subseteq [N, N + B]$. For any integer $r \ge 1$, for the sum*

$$S_\chi(\mathcal{A}, \mathcal{B}) = \sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \chi(a + b)$$

with a non-principal multiplicative character modulo a prime p, we have

$$\begin{aligned}S_\chi(\mathcal{A}, \mathcal{B}) \ll A^{1/2}(\#\mathcal{A})^{1/2}\#\mathcal{B}\left(\frac{A + Bp^{1/2r}}{A^2\#\mathcal{B}}\right)^{1/4r} & p^{1/8r+o(1)} \\ &+(\#\mathcal{A})^{1/2}(\#\mathcal{B})^{1/2}\left(A + p^{1/2r}B\right)^{1/2},\end{aligned}$$

where the implied constant may depend only on r.

Proof. We follow the argument on [13, Page 371] and denote by $\nu(u)$ the number of solutions to the congruence

$$a(b_1 - b_2)^{-1} \equiv u \pmod p$$

in integers a, b_1, b_2 with $|a - M - N| < 2A$ and $b_1, b_2 \in \mathcal{B}$, $b_1 \not\equiv b_2 \pmod p$.

Lemma 8 yields the following improvement of [13, Bounds (10)]:

$$\sum_{u=0}^{p-1} \nu(u)^2 \ll A\,(\#\mathcal{B})^3\,p^{o(1)} \tag{38}$$

(instead of $AB(\#\mathcal{B})^2p^{o(1)}$ on the right hand side, as in [13]). Indeed, the sum in (38) is equal to the number of solutions of the congruence

$$a_1(b_1 - b_2) \equiv a_2(b_3 - b_4) \pmod p,$$

where a_1, a_2 belong to an interval I of length $4A$, $b_1, b_2, b_3, b_4 \in \mathcal{B}$ and $b_1 \neq b_2, b_3 \neq b_4$. Furthermore $\mathcal{B}$ belongs to an interval of length B. We fix $b_2, b_4 \in \mathcal{B}$ and thus get that the number of solutions is less than $(\#\mathcal{B})^2$ times the number of solutions of

$$a_1\widetilde{b}_1 \equiv a_2\widetilde{b}_2 \pmod p, \tag{39}$$

where $a_1, a_2 \in I$ and $\widetilde{b}_1, \widetilde{b}_2 \neq 0$ belong to the union of two "shifted" sets $\mathcal{B} - b_2$ and $\mathcal{B} - b_4$, respectively. Applying Lemma 8 we derive the bound

$$((\#\mathcal{B})^2 + A\#\mathcal{B})p^{o(1)} < A\#\mathcal{B}p^{o(1)}$$

(since $B < A$) on the number of solutions to (39), which in turn yields (38).

This yields instead of [13, Bound (11)], the bound

$$S_\chi(\mathcal{A}, \mathcal{B}) \ll (\#\mathcal{A})^{1/2}\,(\#\mathcal{B})^{1-1/4r}\,A^{1/4-1/4r}(A + p^{1/2r}B)^{1/4}p^{1/8r+o(1)} + (A\#\mathcal{A}\#\mathcal{B})^{1/2}.$$

Concluding the argument as in [13] we obtain the result. □

Taking $r = 2$, we derive:

Corollary 23. *Under the conditions of Theorem 22 if $\mathcal{A} = \mathcal{B}$, $A \le p^{1/2}$ and $\#\mathcal{A} > p^{9/20+\varepsilon}$ for some $\varepsilon > 0$, then we have*

$$S_\chi(\mathcal{A}, \mathcal{A}) \ll (\#\mathcal{A})^2 p^{-\delta},$$

where $\delta > 0$ depends only on ε.

We recall that it has been noted in [12, Remark] that the bound (38) and Corollary 23 hold under some additional conditions. So here we recover the same estimates without that restriction.

12. Character Sums with the Divisor Function

Next, we consider the sum

$$S_a(N) = \sum_{1\le n\le N} \tau(n)\chi(a+n), \tag{40}$$

where $a \in \mathbb{Z}$, τ is the divisor function and χ is a non-principal multiplicative character modulo a prime p.

Karatsuba [17] has established a non-trivial estimate for (40) uniformly over the integers a with $\gcd(a,p) = 1$ provided that $N \ge p^{1/2+\varepsilon}$ with some fixed $\varepsilon > 0$. Chang [7] has extended this result to $N \ge p^{\rho+\varepsilon}$ where

$$\rho = \frac{1}{8}(7-\sqrt{17}) = 0.359\ldots.$$

Furthermore, Karatsuba [18] has also shown that if $0 < |a| \le p^{1/2}$ then the sums (40) can be nontrivially estimated already for $N \ge p^{1/3+\varepsilon}$. Here we show that one has a nontrivial estimate of $S_a(N)$ for $N \ge p^{1/3+\varepsilon}$ and any integer a with $\gcd(a,p) = 1$.

Theorem 24. *For any $\varepsilon > 0$ there is $\delta > 0$ such that if $N \ge p^{1/3+\varepsilon}$ then uniformly over the integers a with* $\gcd(a,p) = 1$*,*

$$S_a(N) \ll Np^{-\delta}.$$

Proof. We can assume that $N < p^{1/2+0.1\varepsilon}$ since otherwise the result follows from the aforementioned result of Karatsuba [17]. Let $X_0 = \sqrt{N}$. We have

$$\begin{aligned} S_a(N) &= \sum_{1\le x\le X_0} \chi(a+x^2) + \sum_{1\le x\le X_0} 2 \sum_{x<y\le N/x} \chi(a+xy) \\ &= 2\sigma + O(\sqrt{N}), \end{aligned} \tag{41}$$

where

$$\sigma = \sum_{1\le x< X_0} \sum_{x<y\le N/x} \chi(a+xy).$$

Now we split the sum W into $L = \lfloor (\log X_0)/\log 2\rfloor$ sums

$$\sigma_j = \sum_{2^{j-1}\le x<\min(X_0,2^j)} \sum_{x<y\le N/x} \chi(a+xy), \quad j = 1,\ldots,L.$$

We specify j so that for $\widetilde{\sigma} = \sigma_j$ we have

$$|\sigma| \ll |\widetilde{\sigma}|L \ll |\widetilde{\sigma}| \log p \tag{42}$$

and define

$$I = \{x : 2^{j-1} \le x < \min(X_0, 2^j)\}.$$

Furthermore, let

$$\eta = \varepsilon/4, \qquad Z = Np^{-2\eta}2^{-j}, \qquad T = \lfloor p^\eta \rfloor$$

and let $\mathcal{P}$ be the set of the primes $z \in (Z/2, Z]$. Following [18], we observe that

$$\widetilde{\sigma} = \Sigma + O(Np^{-\eta}), \tag{43}$$

where

$$\Sigma = \frac{1}{\#\mathcal{P}T} \sum_{x\in I} \sum_{x<y\le N/x} \sum_{z\in\mathcal{P}} \sum_{t=1}^{T} \chi(a + x(y+zt)).$$

We now prove that for a sufficiently large p we have

$$|\Sigma| \le Np^{-2\delta} \tag{44}$$

for some $\delta > 0$ that depends only on ε. Then the desired result follows from (41), (42) and (43).

Defining

$$S(x,y,z) = \sum_{t=1}^{T} \chi(a + x(y+zt))$$

we have

$$|\Sigma| \le \frac{1}{\#\mathcal{P}T} \sum_{x\in I} \sum_{x<y\le N/x} \sum_{z\in\mathcal{P}} |S(x,y,z)|. \tag{45}$$

Assume that (44) does not hold and define $\mathcal{E}$ as the set of triples (x,y,z) involved in the summation in (45) and such that $|S(x,y,z)| \ge Tp^{-3\delta}$. Then we have

$$\#\mathcal{E} \ge N\#\mathcal{P}p^{-3\delta}$$

provided that p is large enough. Using the multiplicativity of χ we derive

$$|S(x,y,z)| = \left| \sum_{t=1}^{T} \chi(ax^{-1}z^{-1} + yz^{-1} + t) \right|,$$

where x^{-1}, z^{-1} are considered in $\mathbb{F}_p$, and define

$$\mathcal{U} = \{ax^{-1}z^{-1} + yz^{-1} \ : \ (x,y,z) \in \mathcal{E}\}.$$

Thus, we get

$$\left| \sum_{t=1}^{T} \chi(u+t) \right| \ge Tp^{-3\delta}$$

for any $u \in \mathcal{U}$. Hence

$$\sum_{u\in U} \left| \sum_{t=1}^{T} \chi(u+t) \right| \ge \#\mathcal{U}Tp^{-3\delta}. \tag{46}$$

Take $X = 2^j$, $Y = N2^{1-j}$. Since we have assumed that $N \le p^{1/2+0.1\varepsilon}$, we have $X^2YZ < p$ provided that p is large enough, so we can use Lemma 9. Hence, we conclude that the congruence

$$ax_1^{-1}z_1^{-1} + y_1z_1^{-1} \equiv ax_2^{-1}z_2^{-1} + y_2z_2^{-1} \pmod p$$

has at most $N\#\mathcal{P}p^{o(1)}$ solutions in $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{E}$. Therefore, recalling that $\eta = \varepsilon/4$ and assuming that $\delta < \varepsilon/14$, we obtain

$$\begin{aligned}\#\mathcal{U} &\ge (N\#\mathcal{P}p^{-3\delta})^2(N\#\mathcal{P}p^{o(1)})^{-1} \gg NZp^{-7\delta} = N^2X^{-1}p^{-2\eta}p^{-7\delta}\\ &\gg N^{3/2}p^{-2\eta}p^{-7\delta} \ge p^{1/2+3\varepsilon/2-2\eta-7\delta} = p^{1/2+\varepsilon-7\delta} \ge p^{1/2+\varepsilon/2}.\end{aligned}$$

Therefore, by a result of Karatsuba (see, for example, [19, p. 52]) we have

$$\sum_{u\in\mathcal{U}} \left| \sum_{t=1}^{T} \chi(u+t) \right| \le \#\mathcal{U}Tp^{-\kappa},$$

where $\kappa > 0$ depends only on ε. Taking $\delta < \min\{\kappa/3, \varepsilon/14\}$ we see that (46) is false, which concludes the proof. □

Remark 25. *Let $\tau_k(n)$ be the number of ordered representations $n = d_1 \dots d_k$ with positive integers $d_1, \dots, d_k$. Our argument can also be used to improve the range of a of the result of* [18] *on analogues of the sums $S_a(N)$ with τ_k instead of τ.*

Acknowledgement

The research of J. B. was partially supported by National Science Foundation Grant DMS-0808042, that of S. V. K. was partially supported by Russian Fund for Basic Research, Grant N. 11-01-00329 and that of I. E. S. by Australian Research Council Grant DP1092835.

References

[1] A. Ayyad, T. Cochrane and Z. Zheng, 'The congruence $x_1x_2 \equiv x_3x_4 \pmod p$, the equation $x_1x_2 = x_3x_4$ and the mean value of character sums', *J. Number Theory*, **59** (1996), 398–413.

[2] U. Betke, M. Henk and J. M. Wills, 'Successive-minima-type inequalities', *Discr. Comput. Geom.*, **9** (1993), 165–175.

[3] J. Bourgain, 'More on the sum-product phenomenon in prime fields and its applications', *Intern. J. Number Theory*, **1** (2005), 1–32.

[4] J. Bourgain, M. Z. Garaev, S. V. Konyagin and I. E. Shparlinski, 'On the hidden shifted power problem', *Preprint*, 2011 (available from `http://arxiv.org/abs/1110.0812`).

[5] T. H. Chan and I. E. Shparlinski, 'On the concentration of points on modular hyperbolas and exponential curves', *Acta Arith.*, **142** (2010), 59–66.

[6] M.-C. Chang, 'Factorization in generalized arithmetic progressions and applications to the Erdős-Szemerédi sum-product problems', *Geom. and Funct. Anal.*, **13** (2003), 720–736.
[7] M.-C. Chang, 'Character sums in finite fields', Amer. Math. Soc., Providence, RI, 2010., 83–98.
[8] M.-C. Chang, J. Cilleruelo, M. Z. Garaev, J. Hernández, I. E. Shparlinski and A. Zumalacárregui, 'Concentration of points and isomorphism classes of hyperelliptic curves over a finite field in some thin families' *Preprint*, 2011 (available from `http://arxiv.org/abs/1111.1543`).
[9] J. Cilleruelo and M. Z. Garaev, 'Concentration of points on two and three dimensional modular hyperbolas and applications', *Geom. and Func. Anal.*, **21** (2011), 892–904.
[10] T. Cochrane and S. Shi, 'The congruence $x_1x_2 \equiv x_3x_4 \pmod p$ and mean values of character sums', *J. Number Theory*, **130** (2010), 767–785.
[11] T. Cochrane and Z. Zheng, 'High order moments of character sums', *Proc. Amer. Math. Soc.*, **126** (1998), 951–956.
[12] J. B. Friedlander and H. Iwaniec, 'The divisor problem for arithmetic progressions', *Acta Arith.*, **45** (1985), 273–277.
[13] J. B. Friedlander and H. Iwaniec, 'Estimates for character sums', *Proc. Amer. Math. Soc.*, **119** (1993), 365–372.
[14] M. Z. Garaev, 'On multiplicative congruences', *Math. Zeitschrift*, (to appear).
[15] M. Z. Garaev and V. Garcia, 'The equation $x_1x_2 = x_3x_4 + \lambda$ in fields of prime order and applications', *J. Number Theory*, **128** (2008), 2520–2537.
[16] J. von zur Gathen and J. Gerhard, *Modern computer algebra*, Cambridge University Press, Cambridge, 2003.
[17] A. A. Karatsuba, 'A certain arithmetic sum', *Doklady Acad. Sci. USSR (Transl. as Soviet Math. Dokl.)*, **199** (1971), 770–772 (in Russian),
[18] A. A. Karatsuba, 'Weighted character sums', *Izv. Ross. Akad. Nauk Ser. Mat. (Transl. as Izv. Math.)*, **64**(2) (2000), 29–42 (in Russian).
[19] A. A. Karatsuba, 'Arithmetic problems in the theory of Dirichlet characters', *Uspekhi Mat. Nauk. (Transl. as Russian Math. Surveys)*, **63**(4) (2008), 43–92 (in Russian).
[20] S. V. Konyagin, 'Bounds of character sums in finite fields', *Matem. Zametki*, **88** (2010), 529–542 (in Russian).
[21] P. Le Boudec, 'Power-free values of the polynomial $t_1 \ldots t_r - 1$', *Bull. Aust. Math. Soc.* (to appear).
[22] M. Mignotte, *Mathematics for computer algebra*, Springer-Verlag, Berlin, 1992.
[23] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Polish Sci. Publ., Warszawa, 1990.
[24] I. E. Shparlinski, 'On the distribution of points on multidimensional modular hyperbolas', *Proc. Japan Acad. Sci., Ser.A*, **83** (2007), 5–9.
[25] I. E. Shparlinski, 'On a generalisation of a Lehmer problem', *Math. Zeitschrift*, **263** (2009), 619–631.
[26] T. Tao and V. Vu, *Additive combinatorics*, Cambridge Stud. Adv. Math., **105**, Cambridge University Press, Cambridge, 2006.

Institute for Advanced Study, Princeton, NJ 08540, USA
E-mail address: bourgain@ias.edu

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, C.P. 58089, Morelia, Michoacán, México
E-mail address: garaev@matmor.unam.mx

Steklov Mathematical Institute, 8, Gubkin Street, Moscow, 119991, Russia
E-mail address: konyagin@mi.ras.ru

Department of Computing, Macquarie University, Sydney, NSW 2109, Australia
E-mail address: igor.shparlinski@mq.edu.au